Math 2010 Week 11

Second Derivative Test

Last time: Definiteness of sym matrices.

Thm If $\Omega \subseteq \mathbb{R}^n$ is open, $f: \Omega \to \mathbb{R}$ is C^2, $a \in \Omega$ is a critical point (i.e. $\nabla f(a) = 0$)

Then $Hf(a)$ is

$$\begin{cases} \text{positive definite} \Rightarrow a \text{ is a local min} \\ \text{negative definite} \Rightarrow a \text{ is a local max} \\ \text{indefinite} \qquad\quad \Rightarrow a \text{ is a saddle point} \end{cases}$$

Idea of Pf Taylor Thm, $\nabla f(a) = 0$

$\Rightarrow$ For x near a,

$$f(x) - f(a) \approx \frac{1}{2}(x-a)^T Hf(a)(x-a)$$

If $Hf(a)$ is positive definite

R.H.S. > 0 for all $x \neq a$

$\Rightarrow f(x) - f(a) > 0$ for all $x \neq a$ and near a.

$\Rightarrow f$ has a local min at a.

"Proof" is similar for the other two cases.

Geometrically

① $Hf(a)$ is positive definite

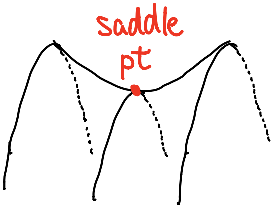
(eg. $f = x^2 + y^2$ at $(0,0)$)

② $Hf(a)$ is negative definite

(eg. $f = -x^2 - y^2$ at $(0,0)$)

③ $Hf(a)$ is indefinite

$\left(\begin{array}{c} \text{eg. } f = x^2 - y^2 \\ \text{at } (0,0) \end{array}\right)$



Next: Determine definiteness of $Hf(a)$

For the simple case $n=2$, it can be done easily by completing square.

Thm Let $M = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$. Then

- M is positive definite $\Leftrightarrow$ $AC - B^2 > 0$, $A > 0$
- M is negative definite $\Leftrightarrow$ $AC - B^2 > 0$, $A < 0$
- M is indefinite $\Leftrightarrow$ $AC - B^2 < 0$

Rmk $AC - B^2 = \det M$

Pf

Let $q(x,y) = [x \ y] M \begin{bmatrix} x \\ y \end{bmatrix} = Ax^2 + 2Bxy + Cy^2$

Case I $(A \neq 0)$

$$Aq(x,y) = A^2x^2 + 2ABxy + ACy^2$$
$$= (Ax + By)^2 + (AC - B^2)y^2$$

Clearly

$q(x,y) > 0 \ \forall (x,y) \neq (0,0) \Leftrightarrow AC - B^2 > 0, \ A > 0$

$q(x,y) < 0 \ \forall (x,y) \neq (0,0) \Leftrightarrow AC - B^2 > 0, \ A < 0$

$q(x,y)$ change signs $\Leftrightarrow AC - B^2 < 0$

Case II $(A = 0)$ $\quad AC - B^2 = -B^2 \leq 0$

$q(x,y) = 2Bxy + Cy^2 = y(2Bx + Cy)$

Clearly q is neither positive or negative definite and is indefinite $\Leftrightarrow B \neq 0 \Leftrightarrow AC - B^2 < 0$

Thm (Second Derivative Test)

If $\Omega \subseteq \mathbb{R}^2$ is open, $f : \Omega \to \mathbb{R}$ is C^2, $a \in \Omega$, $\nabla f(a) = 0$. Then

① $f_{xx}f_{yy} - f_{xy}^2 > 0$, $f_{xx} > 0$ at $a \Rightarrow a$ is a local min.

② $f_{xx}f_{yy} - f_{xy}^2 > 0$, $f_{xx} < 0$ at $a \Rightarrow a$ is a local max.

③ $f_{xx}f_{yy} - f_{xy}^2 < 0$ at $a \Rightarrow a$ is a saddle point

④ $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $a \Rightarrow$ inconclusive.

Rmk
- $f_{xx}f_{yy} - f_{xy}^2 = \det Hf$
- For ④, a can be local max/min or saddle pt (see eg 3)

eg 1 $f(x,y) = 3x^2 - 10xy + 3y^2 + 2x + 2y + 3$

Find and classify critical points of f

Sol f is polynomial, so is differentiable on $\mathbb{R}^2$

$$\nabla f = [f_x \ \ f_y] = [6x - 10y + 2 \quad -10x + 6y + 2]$$

$$\nabla f = \vec{0} \iff \begin{cases} 6x - 10y + 2 = 0 \\ -10x + 6y + 2 = 0 \end{cases}$$

$$\iff (x,y) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$\therefore \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$ is the only critical point.

$$Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6 & -10 \\ -10 & 6 \end{bmatrix}$$

$$f_{xx}f_{yy} - f_{xy}^2 = (6)^2 - (-10)^2 = -64 < 0$$

$$f_{xx} = 6 > 0$$

By 2^{nd} derivative test,

$\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$ is a saddle point eg 2 $f(x,y) = 3x - x^3 - 3xy^2$

Find and classify critical points of f

Sol f is a polynomial,

so is differentiable on $\mathbb{R}^2$

$\nabla f = [f_x \quad f_y]$

$= [3-3x^2-3y^2 \quad -6xy]$

$\nabla f = 0$

$$\iff \begin{cases} 3-3x^2-3y^2 = 0 \cdots (1) \\ -6xy = 0 \cdots (2) \end{cases}$$

(2) $\Rightarrow$ $x=0$ or $y=0$

If $x=0$, (1) $\Rightarrow 3-3y^2=0 \Rightarrow y=\pm 1$

If $y=0$, (1) $\Rightarrow 3-3x^2=0 \Rightarrow x=\pm 1$

4 critical points : $(0,\pm 1)$, $(\pm 1, 0)$

$$Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -6x & -6y \\ -6y & -6x \end{bmatrix}$$

Critical point a	$Hf(a)$	$\det Hf(a)$ $f_{xx}f_{yy}-f_{xy}^2$	$f_{xx}(a)$	Nature of a
$(0,1)$	$\begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix}$	$-36<0$	No need to check	Saddle point
$(0,-1)$	$\begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}$	$-36<0$	No need to check	Saddle point
$(1,0)$	$\begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$	$36>0$	$-6<0$	local max
$(-1,0)$	$\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$	$36>0$	$6>0$	local min

eg 3 (Inconclusive from 2^{nd} derivative test)

$f(x,y) = x^2+y^4$ $\quad$ $g(x,y) = x^2-y^4$ $\quad$ $h(x,y) = -x^2-y^4$

$\nabla f = [2x \;\; 4y^3]$ $\quad$ $\nabla g = [2x \;\; -4y^3]$ $\quad$ $\nabla h = [-2x \;\; -4y^3]$

$\Rightarrow$ $(0,0)$ is a critical point of f, g, h

$Hf = \begin{bmatrix} 2 & 0 \\ 0 & 12y^2 \end{bmatrix}$ $\quad$ $Hg = \begin{bmatrix} 2 & 0 \\ 0 & -12y^2 \end{bmatrix}$ $\quad$ $Hh = \begin{bmatrix} -2 & 0 \\ 0 & -12y^2 \end{bmatrix}$

$Hf(0,0) = Hg(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad$ $Hh(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ Each Hessian matrix has zero determinant at $(0,0)$

2^{nd} derivative test is inconclusive.

Rmk Clearly, f, g, h has local min, saddle point and local max at $(0,0)$ respectively

Second Derivative Test for general n

Let $f: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be C^2, $a \in \Omega$, $\nabla f(a) = 0$.

$$Hf(a) = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & & \ddots & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{bmatrix}$$

f is $C^2 \Rightarrow Hf(a)$ is symmetric. By linear algebra,

$\exists$ orthogonal $n \times n$ matrix P (i.e. $P^TP = I_n$) s.t.

$$P^T Hf(a) P = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

where λ_i are eigenvalues of $Hf(a)$. Hence,

$$Hf(a) \text{ is } \begin{cases} \text{positive definite} \iff \text{All } \lambda_i > 0 \\ \text{negative definite} \iff \text{All } \lambda_i < 0 \\ \text{indefinite} \iff \text{Some } \lambda_i > 0, \text{ some } \lambda_j < 0 \end{cases}$$

Another way to check definiteness of $Hf(a)$

Let H_k be the k by k submatrix

$$H_k = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_k} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_k} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_kx_1} & f_{x_kx} & \cdots & f_{x_kx_k} \end{bmatrix}$$

① $Hf(a)$ is positive definite

$\iff \det H_k > 0$ for $k = 1, 2, \cdots, n$

② $Hf(a)$ is negative definite

$$\iff \det H_k \begin{cases} < 0 & \text{if } k \text{ is odd} \\ > 0 & \text{if } k \text{ is even} \end{cases}$$

For $n = 2$,

$$\det H_1 = \det [f_{xx}] = f_{xx}$$

$$\det H_2 = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

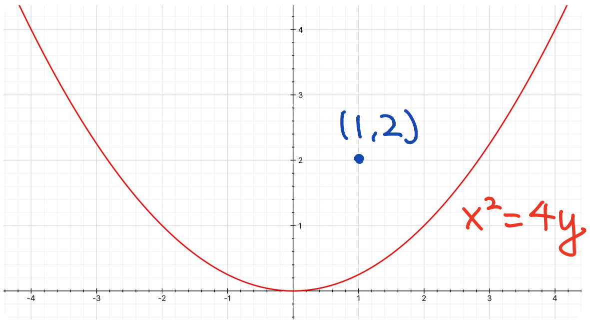
Same result as before.

Lagrange Multipliers

Find Extrema under constraints

eg1 Find the point on the parabola

$x^2 = 4y$ closest to $(1,2)$



$\Updownarrow$

Find minimum of $f(x,y) = (x-1)^2 + (y-2)^2$

under constraint $g(x,y) = x^2 - 4y = 0$

expressed as a level set $g=0$

Theorem (Lagrange Multipliers)

Let f, g be C^1 functions on $\Omega \subseteq \mathbb{R}^n$

$S = g^{-1}(c) = \{x \in \Omega : g(x) = c\}$

Suppose ① a is a local extremum of f on S

② $\nabla g(a) \neq 0$

Then $\begin{cases} \nabla f(a) = \lambda \nabla g(a) \text{ for some } \lambda \in \mathbb{R} \\ g(a) = c \end{cases}$

Rmk ① λ is called Lagrange Multiplier

② Let $F(x,\lambda) = f(x) - \lambda(g(x) - c)$

Then $\nabla F(x,\lambda) = (\underbrace{\nabla f(x) - \lambda g(x)}_{n \text{ components}}, g(x) - c)$

Find critical pt of f under constraint $g=c$ $\Leftrightarrow$ Find critical pt of F without constraint

Back to our example,

Minimize $f(x,y) = (x-1)^2 + (y-2)^2$

Constraint. $g(x,y) = x^2 - 4y = 0$

Sol

f, g are differentiable on $\mathbb{R}^2$

$\nabla f = [\, 2(x-1) \quad 2(y-2) \,]$

$\nabla g = [\, 2x \quad -4 \,] \neq \vec{0}$ on $\mathbb{R}^2$

Suppose (x,y) is a local extremum of $f(x,y)$ on $g(x,y)=0$.

Then by Lagrange Multipliers

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \text{ for some } \lambda \in \mathbb{R} \\ g(x,y) = 0 \end{cases}$$

$$\begin{cases} 2(x-1) = 2\lambda x & \cdots ① \\ 2(y-2) = -4\lambda & \cdots ② \\ x^2 - 4y = 0 & \cdots ③ \end{cases}$$

① $\Rightarrow x - 1 = \lambda x \Rightarrow x(1-\lambda) = 1$

② $\Rightarrow y - 2 = -2\lambda \Rightarrow y = 2(1-\lambda) = \frac{2}{x}$

③ $\Rightarrow x^2 - \frac{8}{x} = 0, \quad x^3 - 8 = 0 \Rightarrow x = 2$

$\therefore y = \frac{2}{2} = 1$

Easy to check $(x,y) = (2,1)$ is a solution

Geometrically, f must have a min. on $g = 0$

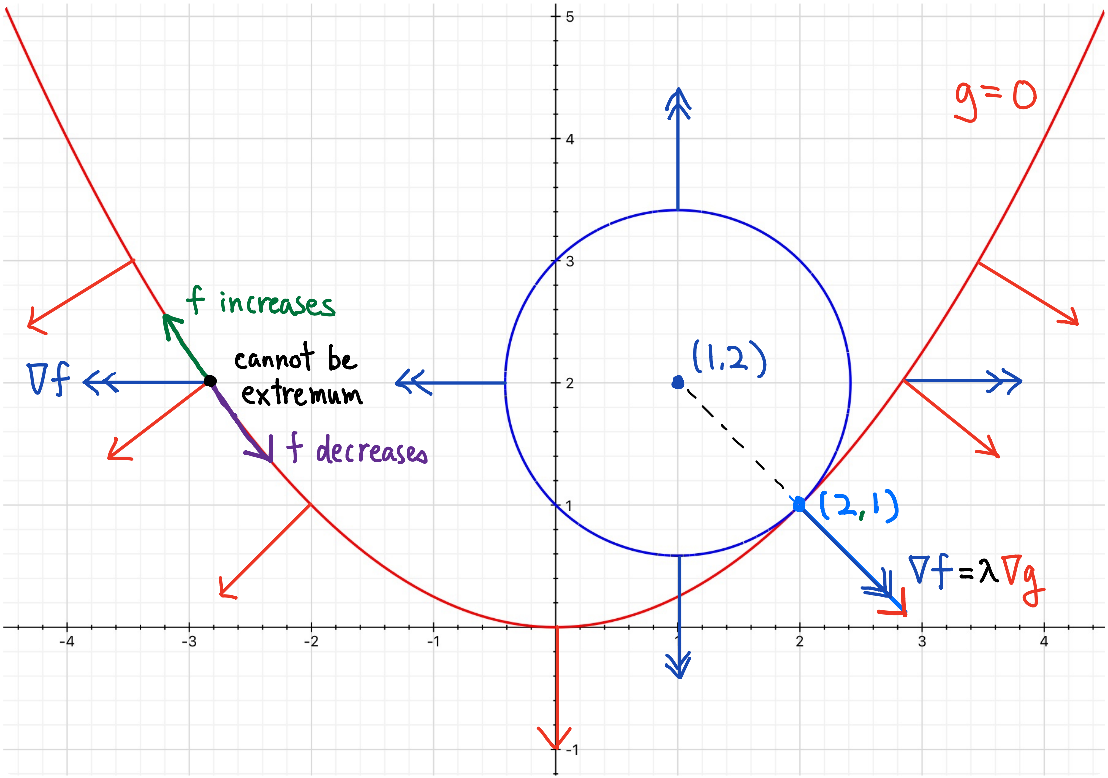
By Lagrange Multipliers, only one pt can be that min.

$\Rightarrow f$ has minimum at $(2,1)$ on $g = 0$.

$f(x,y) = (x-1)^2 + (y-2)^2$ $\longrightarrow$ direction of ∇g ($\perp$ level set of g)

$g(x,y) = x^2 - 4y$ $\longrightarrow$ direction of ∇f (direction which f increases most rapidly)

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$$



Ex Find the point on the parabola $x^2 = 4y$ closest to $(2,5)$

$f(x,y) = (x-2)^2 + (y-5)^2$

$g(x,y) = x^2 - 4y$

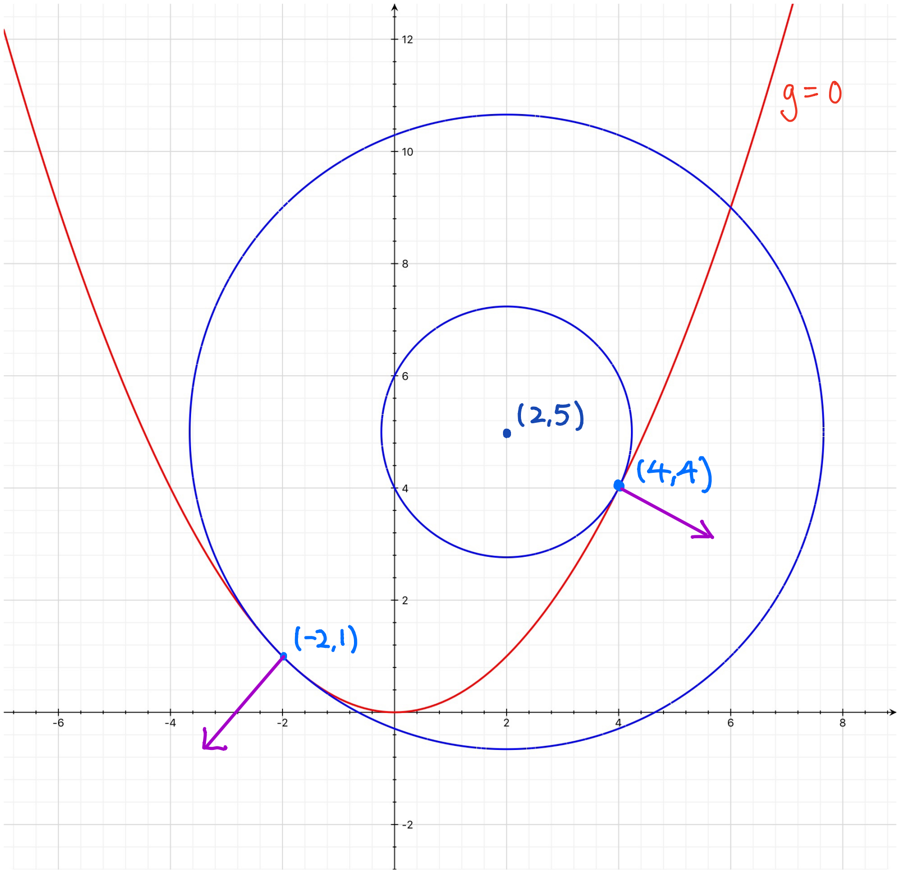
Rmk

$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$ has solutions

$(4,4)$, $(-2,1)$

↑ global min on $g=0$

↑ Not local extremum on $g=0$



eg2 Maximize xy^2 on the ellipse

$$x^2 + 4y^2 = 4$$

Sol Let $f(x,y) = xy^2$

$$g(x,y) = x^2 + 4y^2$$

Note f is continuous and the ellipse $g=4$ is closed and bounded

By EVT,

f has global max and min. on $g=4$

$$\nabla f = [\, y^2 \quad 2xy \,]$$

$$\nabla g = [\, 2x \quad 8y \,]$$

Note $\nabla g \neq 0$ on $x^2+4y^2=4$

Lagrange Multipliers:

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 4 \end{cases} \Leftrightarrow \begin{cases} y^2 = 2\lambda x & \cdots (1) \\ 2xy = 8\lambda y & \cdots (2) \\ x^2+4y^2 = 4 & \cdots (3) \end{cases}$$

Case 1 If $y=0$, then (3) $\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$, $\lambda = 0$ by (1)

$\therefore (x,y) = (\pm 2, 0)$

Case 2 If $y \neq 0$, then

$$\frac{(2)}{(1)} \Rightarrow \frac{2xy}{y^2} = \frac{8\lambda y}{2\lambda x} \Rightarrow \frac{2x}{y} = \frac{4y}{x} \Rightarrow x^2 = 2y^2$$

By (3), $6y^2 = 4 \Rightarrow y = \pm\sqrt{\frac{2}{3}}$

$\therefore x^2 = 2y^2 = \frac{4}{3} \Rightarrow x = \pm\sqrt{\frac{4}{3}}$

$\therefore (x,y) = \left(\pm\sqrt{\frac{4}{3}}, \pm\sqrt{\frac{2}{3}}\right)$

Comparing values of f at the 6 points found using Lagrange Multipliers.

$$f(x,y) = xy^2$$

$$f(\pm 2, 0) = 0$$

$$f\left(\sqrt{\tfrac{4}{3}}, \pm\sqrt{\tfrac{2}{3}}\right) = \sqrt{\tfrac{4}{3}} \cdot \tfrac{2}{3} = \frac{4}{3\sqrt{3}} \leftarrow \text{max}$$

$$f\left(-\sqrt{\tfrac{4}{3}}, \pm\sqrt{\tfrac{2}{3}}\right) = -\sqrt{\tfrac{4}{3}} \cdot \tfrac{2}{3} = -\frac{4}{3\sqrt{3}} \leftarrow \text{min}$$

For $f(x,y)$ on $g=4$,

Global max value $= \dfrac{4}{3\sqrt{3}}$ at $\left(\sqrt{\tfrac{4}{3}}, \pm\sqrt{\tfrac{2}{3}}\right)$

Global min value $= -\dfrac{4}{3\sqrt{3}}$ at $\left(-\sqrt{\tfrac{4}{3}}, \pm\sqrt{\tfrac{2}{3}}\right)$

<u>Rmk</u> We may use another form of Lagrange Multiplier

Let
$$F(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - 4)$$
$$= xy^2 - \lambda(x^2 + 4y^2 - 4)$$

$$\nabla F = (y^2 - 2\lambda x,\ 2xy - 8\lambda y,\ x^2 + 4y^2 - 4)$$

$$\nabla F = 0 \iff \begin{cases} y^2 - 2\lambda x = 0 \\ 2xy - 8\lambda y = 0 \\ x^2 + 4y^2 - 4 = 0 \end{cases}$$

Same system as before

For problems of finding max/min of $f: A \to \mathbb{R}$, Lagrange Multipliers can be used to study f on ∂A

Redo an example before:

eg Find global max/min of

$f(x,y) = x^2 + 2y^2 - x + 3$ for $x^2+y^2 \le 1$

Sol Domain $= A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\}$

As found before, f has only one critical point $(\frac{1}{2}, 0)$ in $\mathrm{int}(A)$ with

$$f(\tfrac{1}{2}, 0) = \frac{11}{4}$$

To study f on $\partial A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$ by Lagrange Multipliers:

Let $g(x,y) = x^2 + y^2$

$\nabla g = (2x, 2y) \neq \vec{0}$ on ∂A $(g=1)$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 1 \end{cases} \iff \begin{cases} 2x - 1 = 2\lambda x & \cdots ① \\ 4y = 2\lambda y & \cdots ② \\ x^2 + y^2 = 1 & \cdots ③ \end{cases}$$

② $\Rightarrow (4 - 2\lambda) y = 0$

$\Rightarrow \lambda = 2$ or $y = 0$

By ①, $2x - 1 = 4x$

$x = -\frac{1}{2}$

By ③, $y = \pm \frac{\sqrt{3}}{2}$

By ③, $x = \pm 1$

Comparing the values of f at these points

$f(\frac{1}{2},0) = \frac{11}{4}$

$f(-\frac{1}{2},\frac{\sqrt{3}}{2}) = f(-\frac{1}{2},-\frac{\sqrt{3}}{2}) = \frac{21}{4}$

$f(1,0) = 3$

$f(-1,0) = 5$

Hence, Max value $= \frac{21}{4}$ at $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$

Min value $= \frac{11}{4}$ at $(\frac{1}{2}, 0)$